Wavenumber-2 and Wavenumber- m Vortex Rossby Wave Instabilities in a Generalized Three-Region Model

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ABSTRACT

A recent study using a discrete three-region nondivergent approximation for the inner-core region of a mature hurricane-like vortex fails to catch wavenumber-2 azimuthal shear instabilities that are observed in experiments and predicted by similar continuous model representations.

With hurricane applications in mind, a generalized version of a piecewise uniform three-region vortex model is presented. The necessary and sufficient criteria for wavenumber-2 and $-m$ instabilities are derived and discussed. The peculiar dynamics of vortex Rossby waves on a discrete circular waveguide elucidate why wavenumber-2 instabilities have been difficult to find in previous analyses and also demonstrate some of the idiosyncrasies of the discrete model. The physical structure of the instabilities is also briefly examined.

1. Introduction

The generation, propagation, wave-mean flow, and wave-wave interaction of vortex Rossby waves and their coupling to the boundary layer and cumulus convection is believed important to the inner-core dynamics of hurricanes. Because a hurricane vortex possesses a small height-to-width aspect ratio and a large swirl ratio, we expect its rotational dynamics to be quasi-two-dimensional (Vladimirov and Tarasov 1980, hereafter VT80; Gall 1983). Although the local Rossby radius for convectively forced disturbances in the core region is of the same order as the core's diameter (Shapiro and Montgomery 1993), the strictly two-dimensional nondivergent model still serves as a useful starting place for obtaining basic insight into vortex Rossby wave dynamics arising within the radial-vorticity-gradient waveguide of the azimuthal mean hurricane vortex.

Vortex Rossby waves live on the radial vorticity gradients of the vortex itself. These gradients act as the "waveguides," or the regions where the waves can exist. The barotropic instability processes of these waves have been studied theoretically using a variety of simplifications entailing both continuous and discontinuous waveguide models. While continuous waveguide models are generally regarded as more realistic, their discrete (discontinuous) waveguide counterparts can sometimes clarify the underlying instability mechanisms and dynamics.

A major criticism of the discrete waveguide model for a mature hurricane comprised of a single ring of elevated vorticity in its eyewall, is the frequent lack of a wavenumber-2 exponential instability, especially in discrete three-region models (Michalke and Timme 1967; Rotunno 1978; Schubert et al. 1999, hereafter S99). Wavenumber-2 vorticity instabilities have been observed, however, in experimental studies (VT80; Lugovtsov 1982), quasigeostrophic theoretical studies (Flierl 1988, hereafter F88), observational studies of hurricanes (Kuo 1999; Reasor et al. 2000) and idealized numerical simulations of hurricanes (Zhang et al. 2000; Chen and Yau 2001). Idealized studies of tornadoes and hurricanes employing a continuous representation have also been found to contain these wavenumber-2 instabilities (Staley and Gall 1979; Steffens 1988; Peng and Williams 1991; Montgomery et al. 2000). The effects of diffusion on perturbation vorticity can also destabilize wavenumber-2 modes that are stable in purely inviscid formulations (Nolan and Farrell 1999).

In this paper, we will first show that wavenumber-2 instabilities can exist in a generalization of the barotropic, nondivergent, three-region vortex model. We will derive the necessary and sufficient conditions for the instabilities to occur. We then obtain the criterion for wavenumber- m instability in the generalized framework, which relies on simple ratios of the vortex parameters. Next, we will discuss the structure of these instabilities. During our analysis, we tie this work in with previous instability analyses of the three-region nondivergent model (VT80, F88, S99). Last, we will give the conclusions and note the limitations and peculiarities of the theory.

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2. Two-dimensional stability analysis of a three-region vorticity model

a. Model formulation and dispersion relation

Consider a basic-state circular vortex, defined in cylindrical coordinates (r, λ) , with tangential wind $\bar{v}(r)$. To gain basic understanding of the hurricane's inner-core asymmetric dynamics we assume, for simplicity, strictly two-dimensional nondivergent inviscid vorticity dynamics. After neglecting second-order terms in the equation for material conservation of vorticity, the perturbation streamfunction $\psi'(r, \lambda, t)$ is governed by the linearized barotropic nondivergent vorticity equation (e.g., Weber and Smith 1993)

$$\left(\frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \lambda}\right) \nabla^2 \psi' - \frac{\partial \psi'}{r \partial \lambda} \frac{d\bar{\zeta}}{dr} = 0, \quad (1)$$

where $\bar{\omega}(r) = \bar{v}(r)/r$ is the basic-state angular velocity, $\bar{\zeta}(r) = d(r\bar{v})/r dr$ is the basic-state relative vorticity.

Assuming azimuthally propagating wave solutions of the form $\psi'(r, \lambda, t) = \hat{\psi}(r)e^{i(m\lambda - \nu t)}$, where m is the tangential wavenumber, and ν the complex frequency, one obtains (e.g., Gent and McWilliams 1986)

$$(\nu - m\bar{\omega}) \left[r \frac{d}{dr} \left(r \frac{d\hat{\psi}}{dr} \right) - m^2 \hat{\psi} \right] + mr \frac{d\bar{\zeta}}{dr} \hat{\psi} = 0. \quad (2)$$

We will limit our discussion to integral values of m to satisfy the inherent periodicity requirement in the azimuth. However, for plotting convenience, continuous m will be used.

We next adopt the three-region piecewise constant vorticity model studied by Michalke and Timme (1967), VT80, Lugovtsov (1982), F88, Dritschel (1989), and S99. We employ a more general version of this model, allowing the vorticity to take on any value in any region, including nonzero in the environment. Denoting the environmental vorticity as C , the radial vorticity distribution is given by

$$\bar{\zeta}(r) = \frac{\partial(r\bar{v})}{r \partial r} = \begin{cases} \xi_1 + \xi_2 + C & 0 < r < r_1, \\ \xi_2 + C & r_1 < r < r_2, \\ C & r_2 < r < \infty, \end{cases}$$

where ξ_1 , ξ_2 , r_1 , and r_2 are constants. Respectively, ξ_1 and ξ_2 are the vorticity jumps at the inner and outer interfaces. The tangential wind ($\bar{v}$) and the angular velocity ($\bar{\omega}$) profiles consistent with this vorticity profile are

$$r\bar{v}(r) = \frac{1}{2} \begin{cases} \xi_1 r^2 + \xi_2 r^2 + Cr^2 & 0 \leq r \leq r_1, \\ \xi_1 r_1^2 + \xi_2 r^2 + Cr^2 & r_1 \leq r \leq r_2, \\ \xi_1 r_1^2 + \xi_2 r_2^2 + Cr^2 & r_2 \leq r < \infty. \end{cases}$$

And,

FIG. 1. The generalized three-region model.

$$\bar{\omega}(r) = \frac{1}{2} \begin{cases} \xi_1 + \xi_2 + C & 0 \leq r \leq r_1, \\ \xi_1 \left(\frac{r_1}{r}\right)^2 + \xi_2 + C & r_1 \leq r \leq r_2, \\ \xi_1 \left(\frac{r_1}{r}\right)^2 + \xi_2 \left(\frac{r_2}{r}\right)^2 + C & r_2 \leq r < \infty. \end{cases} \quad (3)$$

For an illustration of this model, see Fig. 1. When discussing this model, we refer to the region $0 \leq r \leq r_1$ as the “inner region,” the region $r_1 \leq r \leq r_2$ as the “annulus,” and the region $r_2 \leq r < \infty$ as the “environment.” Also, the terms “inner interface” and “outer interface” refer to the vorticity jumps at r_1 and r_2 , respectively.

Rayleigh's condition for shear instabilities in the continuous model requires the vorticity gradient to change sign somewhere in the domain (Michalke and Timme 1967; Gent and McWilliams 1986). This condition is satisfied in this discrete model when the jumps ξ_1 and ξ_2 are of different sign. We will therefore assume that ξ_1 and ξ_2 are of opposite sign for the remainder of this paper.

Returning to (1), we follow the analysis of S99 and restrict our attention to vorticity perturbations that arise only from the radial advection of the basic state vorticity. This restriction implies that (1) reduces to $\nabla^2 \psi' = 0$ (irrotational perturbation flow) for $r \neq r_1$ and $r \neq r_2$. Equivalently, (2) becomes

$$r \frac{d}{dr} \left(\frac{d\hat{\psi}}{dr} \right) - m^2 \hat{\psi} = 0 \quad \text{for } r \neq r_1, r_2. \quad (4)$$

The general solution to this ordinary differential equation is

$$\hat{\psi}(r) = \Psi_1 B_1^{(m)}(r) + \Psi_2 B_2^{(m)}(r), \quad (5)$$

where Ψ_1 and Ψ_2 are complex constants and, for $i = 1, 2$,

$$B_i^{(m)}(r) = \begin{cases} (r/r_i)^m & 0 \leq r \leq r_i, \\ (r_i/r)^m & r_i \leq r < \infty. \end{cases} \quad (6)$$

Using this solution and integrating (2) over small intervals around $r = r_1$ and $r = r_2$, one obtains the following eigenvalue problem:

$$\begin{pmatrix} m\bar{\omega}(r_1) - \frac{1}{2}\xi_1 & -\frac{1}{2}\xi_1\left(\frac{r_1}{r_2}\right)^m \\ -\frac{1}{2}\xi_2\left(\frac{r_1}{r_2}\right)^m & m\bar{\omega}(r_2) - \frac{1}{2}\xi_2 \end{pmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} = \nu \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix}. \quad (7)$$

Solving for ν , we get the following expression for the wave frequency (i.e., the dispersion relation):

$$\nu = \frac{1}{2}(\nu_1 + \nu_2) \pm \frac{1}{2}\sqrt{(\nu_1 - \nu_2)^2 + \xi_1\xi_2(r_1/r_2)^{2m}}, \quad (8)$$

where

$$\nu_1 = m\bar{\omega}(r_1) - \frac{1}{2}\xi_1, \quad \nu_2 = m\bar{\omega}(r_2) - \frac{1}{2}\xi_2 \quad (9)$$

are the discrete, noninteracting vortex Rossby wave frequencies at the inner and outer interfaces.

In S99, ν is real for $m = 1, 2$ and thus, these wavenumbers are always exponentially stable. In the generalized model used here, however, we will show that an exponential shear instability can occur for $m = 2$. While confirming previous work by VT80, we extend their theory and explain the mechanism of this instability.

b. Necessary and sufficient conditions

To begin the instability analysis, it is useful and convenient to define additional parameters. Let

$$\delta = r_1/r_2 < 1, \quad \epsilon = -\xi_2/\xi_1 > 0.$$

Next, assume that ν has a nonzero imaginary part. In other words, assume that the discriminant in (8) is negative:

$$(\nu_1 - \nu_2)^2 + \xi_1\xi_2\delta^{2m} < 0. \quad (10)$$

For $m = 1$, it can be shown, using (3) and (9), that $\nu_1 = \frac{1}{2}(\xi_2 + C)$ and $\nu_2 = \frac{1}{2}(\xi_1\delta^2 + C)$. Thus, for the instability to occur,

$$\frac{1}{4}(\xi_2 - \xi_1\delta^2)^2 + \xi_1\xi_2\delta^2 < 0,$$

which can be simplified to

$$(\xi_2 + \xi_1\delta^2)^2 < 0.$$

This proves, by contradiction, that wavenumber-1 is exponentially stable in the generalized model. However, the possibility exists that the three-region model may possess an azimuthal wavenumber-1 algebraic instability, that is, an instability that grows in amplitude al-

gebraically in time (Smith and Rosenbluth 1990; Nolan and Montgomery 2000). This is an interesting question deferred for future work.

Turning our attention to $m = 2$, from (3) and (9), we obtain $\nu_1 = \frac{1}{2}\xi_1 + \xi_2 + C$ and $\nu_2 = \xi_1\delta^2 + \frac{1}{2}\xi_2 + C$. Assuming that we have an instability, (10) gives

$$\left[\left(\frac{1}{2} - \delta^2\right)\xi_1 + \frac{1}{2}\xi_2\right]^2 + \xi_1\xi_2\delta^4 < 0,$$

or, equivalently,

$$(\xi_1 + \xi_2)^2 - 4\xi_1(\xi_1 + \xi_2)\delta^2 + 4\xi_1(\xi_1 + \xi_2)\delta^4 < 0. \quad (11)$$

Because the first term in the left-hand sum is nonnegative, a necessary condition for the inequality to be satisfied is

$$4\xi_1(\xi_1 + \xi_2)\delta^4 < 4\xi_1(\xi_1 + \xi_2)\delta^2, \quad \text{or}$$

$$\xi_1(\xi_1 + \xi_2)\delta^2 < \xi_1(\xi_1 + \xi_2).$$

However, $0 < \delta < 1$, in this model. This implies that $\xi_1(\xi_1 + \xi_2) > 0$. Thus, for an instability to be possible in this model, $|\xi_2| < |\xi_1|$. If the opposite is true, we have stability. This is a necessary, but not sufficient, condition for instability of azimuthal wavenumber-2 disturbances in the three-region model.

For the class of vortices considered in S99 ($C = 0$, $\xi_1 < 0$, $\xi_2 > 0$, that is, compact vortices with elevated vorticity in the annulus), the necessary condition requires negative vorticity in the inner region. In S99 though, the vorticity in the inner region was assumed positive in accordance with numerous flight level observations of cyclonic vorticity in the eye region of hurricanes. Thus, S99's class of vortices lie outside the necessary condition for wavenumber-2 exponential instability. It is precisely this condition that makes wavenumber 2 curiously absent from previous discrete waveguide analyses. Most examined compact vortices with non-negative vorticity everywhere, thus excluding them a priori from supporting this instability. In this generalized model, however, the necessary condition, as well as the assumption of positive vorticity in the inner region can both be fulfilled.

The necessary condition, as well as Rayleigh's condition, then implies that $0 < \epsilon < 1$. We reduce (11) to

$$4(1 - \epsilon)\delta^4 - 4(1 - \epsilon)\delta^2 + (1 - \epsilon)^2 < 0. \quad (12)$$

Equation (12) is quadratic in δ^2 with a positive coefficient on the squared term. Thus, the inequality is satisfied for

$$\frac{1 - \sqrt{\epsilon}}{2} < \delta^2 < \frac{1 + \sqrt{\epsilon}}{2}. \quad (13)$$

This implies that if $|\xi_2| < |\xi_1|$ (necessary condition), and if our annular region has dimensions that fall into the proper ranges determined by the vorticity profile (13), a wavenumber-2 instability will occur. This relation between the ratio of the radii and the ratio of the

vorticity jumps is the sufficient condition for this instability.

A similar result was first attained in VT80, assuming that the vorticity in the inner region is zero. Equation (13) generalizes their result by allowing nonzero vorticity in the inner region.

Despite working with the quasigeostrophic approximation and compact vortices, F88 arrived at similar results in the barotropic limit, which is mathematically isomorphic to the two-dimensional nondivergent model. Using $\Delta = \epsilon/(\epsilon - 1)$ and $b = \delta^{-1}$, Eqs. (12) and (13) are equivalent to F88's Eq. (3.7) and the result in F88's section 3.1.i, respectively.

c. Growth rates

Assuming now an arbitrary m , (8) may be rewritten as

$$\begin{aligned} \nu = & \frac{1}{4}[-\xi_1(1 - m - m\delta^2 + 2m\epsilon - \epsilon) + 2Cm] \\ & \pm \frac{1}{4}\sqrt{(-\xi_1[1 - m + m\delta^2 - \epsilon])^2 - 4\xi_1^2\epsilon\delta^{2m}}, \end{aligned}$$

or equivalently,

$$\nu = \frac{Cm}{2} + \frac{-\xi_1}{4}[(1 - m - m\delta^2 + 2m\epsilon - \epsilon) \mp \sqrt{(1 - m + m\delta^2 + \epsilon)^2 - 4\epsilon\delta^{2m}}]. \quad (14)$$

This is the general criterion for instability. Instability occurs when the discriminant of (14) is less than zero:

$$(1 - m + m\delta^2 + \epsilon)^2 - 4\epsilon\delta^{2m} < 0. \quad (15)$$

This condition is dependent on three parameters: (m , δ , ϵ). Because the two-dimensional model is known to be invariant with respect to uniform rotation, C only appears in the real (rotational) part of the complex frequency ν . Note further that for $m = 2$, (15) reduces to (12). A criterion for the instability of wavenumber- m was previously derived by VT80 in the special case where $\epsilon = 1$:

$$F(m) = [m(1 - x) - 2]^2 - 4x^m < 0,$$

where $x = (r_1/r_2)^2 = \delta^2$. In this special case, $F(m)$ is exactly the same as (15).

Returning to the general case and examining the growth rate, we are only concerned about the positive imaginary part of (14):

$$\nu_i = \frac{|\xi_1|}{4}\sqrt{(1 - m + m\delta^2 + \epsilon)^2 - 4\epsilon\delta^{2m}}. \quad (16)$$

We now make ν_i dimensionless by dividing through by $|\xi_1|$. This is different than S99's formulation, where an average vorticity was used for scaling the growth rate. In this case, the average vorticity adds complexity by making the scaled growth rate equation algebraically dependent on C .

Converting these normalized growth rates into nor-

FIG. 2. Normalized e -folding times for $m = 2, 3, 4$. Note that both $m = 3$ and 4 have an instability when $\epsilon > 1$. To get dimensional e -folding times, divide by $|\xi_1|$.

malized e -folding times, Fig. 2 shows these e -folding times for wavenumbers 2, 3, and 4 in δ - ϵ space. Note that wavenumbers 3 and 4 have a tail that extends to $\epsilon > 1$, which corresponds to the instability discussed by S99. For wavenumber 2, however, there is no tail in the region $\epsilon > 1$. This is a restatement of the fact that the instability for wavenumber 2 is strictly confined to vortices with $\epsilon < 1$. As an example, take $\xi_1 = -10^{-2} \text{ s}^{-1}$, a realistic vorticity jump in a mature hurricane (Black and Marks 1991; Kossin and Schubert 2001, Fig. 1). Thus, 10 units of normalized e -folding time correspond to a true e -folding time of 10^3 s , or about 16.7 min, which corresponds favorably to similar calculations in S99.

d. Vortex Rossby wave interpretation

The description of barotropic instabilities via phase-locked, counter-propagating vortex Rossby waves has proven useful in this model (S99) and other analogous models (Hoskins et al. 1985). As has been shown, vorticity gradients act as the waveguides for the instability, thus, we classify these waves as vortex Rossby waves. This concept proves useful for explaining how the $m =$

FIG. 3. $\text{Re}(v/m)$ for varying values of ϵ and δ , in the frame of reference rotating with the environmental vorticity. Instabilities occur where there is only one solution, indicating a pair of imaginary conjugate solutions. Angular phase speeds are normalized by $-\xi_1/4$.

2 instability is supported by the waveguide. To do this, we require the noninteracting frequencies of these waves. These are obtained by allowing the upper-right and lower-left entries in the interaction matrix (7) to approach zero. Formally, this requires either $\delta \rightarrow 0$ (Rankine vortex) or $m \rightarrow \infty$.

The phase-locked, noninteracting wave solution is obtained by equating ν_1 and ν_2 in (9). If $\nu_1 = \nu_2$, then for any m ,

$$\frac{\xi_2}{\xi_1} = m\delta^2 - (m - 1),$$

or equivalently,

$$\epsilon = (m - 1) - m\delta^2. \quad (17)$$

The dashed curves plotted in Fig. 2 correspond to the solution of (17) for varying m . Note that this locus of points lies entirely within the unstable region of interacting Rossby waves given by (15). We also note that (17) is equivalent to the result of F88 (his section 3.1.iii) although, in that context, the wavenumber was restricted to be greater than 2.

Recalling that $0 < \delta < 1$ implies, using (17), that

FIG. 4. Regions in m - ϵ space where instability occurs, as determined by (15) for various values of δ . Shading indicates instability. Note that for $m = 2$, all plots are unstable for some ϵ interval in $[0, 1]$.

$-1 < \epsilon < m - 1$. This latter inequality implies the following consequences.

- For $m = 1$, the inequality states that for noninteracting waves to phase lock, $\epsilon < 0$. This contradicts the requirement of nonnegative ϵ implied by Rayleigh's condition. Thus, the wavenumber-1 Rossby waves cannot phase lock in this model.
- For $m = 2$, noninteracting waves can phase lock for $0 < \epsilon < 1$. This helps explain why $\epsilon = 1$ is a critical barrier for wavenumber 2.
- For $m > 2$, noninteracting waves can phase lock for $0 < \epsilon < m - 1$. This helps explain why the shaded regions in Fig. 2 asymptote to $\epsilon = m - 1$ as $\delta \rightarrow 0$.

Figure 3 shows the complete phase speeds of the interacting waves, which can be deduced from (14), in the frame of reference rotating with the environmental vorticity as a function of azimuthal wavenumber m , for varying ϵ and δ . Where the curve bifurcates indicates the transition from instability to stability, as the two waves begin moving at different phase speeds and cannot reinforce each other's growth.

Also note the movement of the lesser m -bifurcation point as one changes values of ϵ . For $\epsilon < 1$, this m -bifurcation point is less than 2, while for $\epsilon > 1$, it is greater than 2, showing again that for wavenumber 2, $\epsilon = 1$, is a critical barrier.

Figure 4 shows regions where the instability occurs, as determined by (15) in m - ϵ space for varying values of δ . Note that all the regions' outlines in Fig. 4 intersect at $(m, \epsilon) = (2, 1)$. Thus, for any δ , if $\epsilon < 1$, we enter into the unstable part of the region at $m = 2$.

3. Unstable eigenmode structure

Having now examined the conditions and growth rates of this instability, we focus attention on the eigen-

FIG. 5. Sample plot of ψ' . Dashed contours indicate negative values of ψ' . The shaded region is the annular region. The annular region's vorticity here is assumed to be greater than the inner or outer regions' vorticity. Regions of positive and negative vorticity advections are denoted PVA and NVA, respectively. Example perturbation winds associated with only the marked perturbation are denoted by the arrows.

streamfunction structure of this instability. Recall that Ψ_1 and Ψ_2 are the complex multiplicative constants on the basis functions in the solution for $\hat{\psi}$. Assuming, without loss of generality that $\Psi_1 = 1$ in (7), the first row simplifies to

$$\Psi_2 = \left[-\frac{2}{\xi_1}(\nu - m\bar{\omega}_1) - 1 \right] \delta^{-m}, \quad (18)$$

where ν and $\bar{\omega}_1 = \bar{\omega}(r_1)$ have been defined above in (8) and (3), respectively.

Therefore, substituting (18) in the separable solution for ψ' , we see that

$$\psi'(r, \lambda, t) = \text{Re}\{[B_1^{(m)}(r) + \Psi_2 B_2^{(m)}(r)]e^{i(m\lambda - \nu t)}\}, \quad (19)$$

where Re denotes the real part.

Figure 5 is a plot of ψ' for $\delta = 0.5$ and $\epsilon = 0.875$, a configuration that is unstable in this model for wavenumber 2. We also assume that $\xi_1 < 0$ and $\xi_2 > 0$ to emulate a mature Northern Hemispheric hurricane vortex. The annulus of elevated vorticity is shaded. To interpret this figure, recall that $\zeta' = \nabla^2 \psi'$, and consequently, a negative ψ' lobe corresponds to positive ζ' .

Let us first assume that the waves are noninteracting entities. The positive perturbation vorticity lobe on the marked outer interface induces a perturbation velocity field that advects higher vorticity [positive vorticity advection (PVA)] from the annulus on its southeastern side, and advects lower vorticity [negative vorticity advection (NVA)] from the environment on its north-

western side. This will tend to move the lobe clockwise, opposite the mean cyclonic flow on this interface (i.e., retrograde). On the inner interface, however, a similar analysis of induced flow will tend to propagate the disturbances cyclonically (i.e., prograde). In this way, the waves can potentially phase lock, since the mean tangential flow is faster on the outer interface than the inner interface. [cf. (17)].

No longer assuming noninteraction of the waves, the perturbation flow induced by a wave on one interface can influence the wave on the other interface. This is the essence of (13), which gives the relation between the vorticity profile and the size of the annulus in which this wavenumber-2 instability can occur. This induced flow advects the basic state vorticity in such a way as to reinforce the waves on the other interface. In this indirect manner, the waves interact and aid each other's growth (Hoskins et al. 1985). In effect, this wave-mean flow interaction modifies each wave's phase speed, allowing the waves to remain phase locked even when the noninteracting relation (17) is not satisfied. This interaction then creates the envelope of instability around the solution plotted from (17) and displayed in Fig. 2.

4. Discussion and conclusions

Although a vortex similar to the generalized three-region model of Fig. 1 is not necessarily realistic, our analysis, nonetheless, highlights some important points.

First, wavenumber-2 instabilities are not totally absent from this model, as commonly perceived. Previous analyses of this model find no unstable wavenumber-2 modes (S99) mainly for the reason that the strictly non-negative compact vortex model automatically excludes those vortices that could support these modes. However, as noted before, wavenumber-2 instabilities had previously been found both in special cases (VT80) and in more general quasigeostrophic formulations (F88) of this model. The theory here not only confirmed these works, but also extended them by considering a more general case of their model.

Second, while the assumption of nonzero vorticity in the outer region gives an unbounded tangential wind profile at large radii, the potential problem with extending this knowledge to real world applications can be rectified in the context of discrete models if the generalized model is thought of as a limiting case of a discrete four-region model with zero environmental vorticity. Such a model has been formulated by Kossin et al. (2000, and references therein). When r_3 is large compared to r_1 and r_2 , the interaction matrix in Kossin et al. (2000) approaches the interaction matrix of S99. This implies that the inclusion of this additional interface, if far enough from the other interfaces, does not fundamentally alter our analysis. A complete study of this four-region model and how it might apply to wavenumber-2 and higher instabilities with a depressed central

core of vorticity might prove enlightening in the diagnosis of hurricane inner-core intensity change processes if used in conjunction with a continuous stability formulation.

Third, we would like to echo a quote from Balmforth et al. (2001): "It is irritating that conclusions concerning important physical issues, such as the long-term relaxation of the streamfunction to axisymmetry, seem to turn on fussy mathematical restrictions concerning the smoothness or compactness of the vorticity profiles." While we are not studying the decay of perturbations and the return to axisymmetry, but instead the growth of perturbations, their sentiments echo ours when we study this simple model. Indeed, that is one of the purposes of this work: to point out some more idiosyncrasies of this particular discrete vortex model. While we still gain important, yet basic, understanding of this instability from studying this simple model, we also point out the flaws inherent in it. For example, it is strange to think that an instability as common in continuous models and observed phenomenon would require $\epsilon < 1$ (or, in the case of compact vortices where the environmental vorticity is zero, that the core's vorticity must be negative) as a necessary criterion. Thus, while gaining basic understanding, we note that, realistically, much of this "fussy" mathematics serves only as a guide for examining these instabilities.

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REFERENCES

- Balmforth, N. J., S. G. Llewellyn Smith, and W. R. Young, 2001: Disturbing Vortices. *J. Fluid Mech.*, **426**, 95–133.
- Black, P. G., and F. D. Marks, 1991: The structure of an eyewall mesovortex in Hurricane Hugo (1989). Preprints, *19th Conf. on Hurricanes and Tropical Meteorology*, Miami, FL, Amer. Meteor. Soc., 579–582.
- Chen, Y., and M. K. Yau, 2001: Spiral bands in a simulated hurricane. Part I: Vortex Rossby wave verification. *J. Atmos. Sci.*, **58**, 2128–2145.
- Dritschel, D. G., 1989: On the stabilization of a two-dimensional vortex strip by adverse shear. *J. Fluid Mech.*, **206**, 193–221.
- Flierl, G. R., 1988: On the instability of geostrophic vortices. *J. Fluid Mech.*, **197**, 349–388.
- Gall, R. L., 1983: A linear analysis of the multiple vortex phenomenon in simulated tornadoes. *J. Atmos. Sci.*, **40**, 2010–2024.
- Gent, P. R., and R. T. McWilliams, 1986: The instability of barotropic circular vortices. *Geophys. Astrophys. Fluid. Dyn.*, **35**, 209–233.
- Hoskins, B. J., M. E. McIntyre, and A. W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Meteor. Soc.*, **111**, 877–946.
- Kossin, J. P., and W. H. Schubert, 2001: Mesovortices, polygonal flow patterns, and rapid pressure falls in hurricane-like vortices. *J. Atmos. Sci.*, **58**, 2196–2209.
- , —, and M. T. Montgomery, 2000: Unstable interactions between a hurricane's primary eyewall and a secondary ring of elevated vorticity. *J. Atmos. Sci.*, **57**, 3893–3917.
- Kuo, H.-C., R. T. Williams, and J.-H. Chen, 1999: A possible mechanism for the eye rotation of Typhoon Herb. *J. Atmos. Sci.*, **56**, 1659–1673.
- Lugovtsov, B. A., 1982: Laboratory models of tornado-like vortices. *Topics in Atmospheric and Oceanographic Sciences: Intense Atmospheric Vortices*, L. Bengtsson and J. Lighthill, Eds., Springer-Verlag, 299–312.
- Michalke, A., and A. Timme, 1967: On the inviscid instability of certain two-dimensional vortex-type flows. *J. Fluid Mech.*, **29**, 647–666.
- Montgomery, M. T., J. M. Hidalgo, and P. D. Reasor, 2000: A semi-spectral numerical method of modeling the vorticity dynamics of the near-core region of hurricane-like vortices. Atmospheric Science Paper 695, Colorado State University, 56 pp. [Available from Dept. of Atmospheric Science, Colorado State University, Fort Collins, CO 80523.]
- Nolan, D. S., and B. F. Ferrell, 1999: The intensification of two-dimensional swirling flows by stochastic asymmetric forcing. *J. Atmos. Sci.*, **56**, 3937–3962.
- , and M. T. Montgomery, 2000: The algebraic growth of wave-number-1 disturbances in hurricane-like vortices. *J. Atmos. Sci.*, **57**, 3514–3538.
- Peng, M. S., and R. T. Williams, 1991: Stability analysis of barotropic vortices. *Geophys. Astrophys. Fluid Dyn.*, **58**, 263–283.
- Reasor, P. D., M. T. Montgomery, F. D. Marks, and J. F. Gamache, 2000: Low-wavenumber structure and evolution of the hurricane inner core observed by airborne dual-Doppler radar. *Mon. Wea. Rev.*, **128**, 1653–1680.
- Rotunno, R., 1978: A note on the stability of a cylindrical vortex sheet. *J. Fluid Mech.*, **87**, 761–771.
- Schubert, W. H., M. T. Montgomery, R. K. Taft, T. A. Guinn, S. R. Fulton, J. P. Kossin, and J. P. Edwards, 1999: Polygonal eyewalls, asymmetric eye contraction, and potential vorticity mixing in hurricanes. *J. Atmos. Sci.*, **56**, 1197–1223.
- Shapiro, L. J., and M. T. Montgomery, 1993: A three-dimensional balance theory for rapidly rotating vortices. *J. Atmos. Sci.*, **50**, 3322–3335.
- Smith, R. A., and M. N. Rosenbluth, 1990: Algebraic instability of hollow electron columns and cylindrical vortices. *Phys. Rev. Lett.*, **64**, 649–652.
- Staley, D. O., and R. L. Gall, 1979: Barotropic instability in a tornado vortex. *J. Atmos. Sci.*, **36**, 973–981.
- Steffens, J. L., 1988: The effect of vorticity-profile shape on the instability of a two-dimensional vortex. *J. Atmos. Sci.*, **45**, 254–259.
- Vladimirov, V. A., and V. F. Tarasov, 1980: Formation of a system of vortex filaments in a rotating liquid. *Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza*, **1**, 44–51.
- Weber, H. C., and R. K. Smith, 1993: The stability of barotropic vortices: Implications for tropical cyclone motion. *Geophys. Astrophys. Fluid Dyn.*, **70**, 1–30.
- Zhang, D.-L., Y. Liu, and M. K. Yau, 2000: A multiscale numerical study of Hurricane Andrew (1992). Part III: Dynamically induced vertical motion. *Mon. Wea. Rev.*, **128**, 3772–3788.